CALCULATION OF RADICALS.

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FOR want of room I shall confine my self to the statement of theorems and rules.

1. *Theorem:* Let z and a be any positive numbers and k any positive integer; further let $x = \sqrt[n]{z}$. Form the expression

$$P = a^{n-1} + a^{n-2}x + a^{n-3}x^2 + \ldots + ax^{n-2} + x^{n-1}$$
$$= \frac{a^n - z}{a - x};$$

develop the power P^k in ascending powers of x, and always substitute in this development z for x^n, then the expression for P^k will assume the form

$$P^k = A_{k,\,n-1} + A_{k,\,n-2}x + A_{k,\,n-3}x^2 + \ldots + A_{k\,0}x^{n-1}. \quad \ldots \quad (1)$$

Now the n successive ratios

$$\frac{A_{k,\,n-1}}{A_{k,\,n-2}},\ \frac{A_{k,\,n-2}}{A_{k,\,n-3}},\ \ldots,\ \frac{A_{k\,1}}{A_{k\,0}} \text{ and } \frac{z.A_{k\,0}}{A_{k,\,n-1}}$$

are n different expressions for the real root of n^{th} degree of z with the same degree of approximation; or what amounts to the same, for sup. lim. $k=\infty$

$$\frac{A_{k\,h}}{A_{k,\,h-1}} = \sqrt[n]{z}, \quad \ldots \ldots \quad (2)$$

and superior limit $k=\infty$ $\quad \dfrac{A_{k\,h}}{A_{k,\,h-m}} = \sqrt[n]{z^m}.$

The calculated values of the quantities A, which may be called *components*, are as follows:

Let $A_0, A_1, A_2, \ldots$ be the binomial coefficients in the expansion of $(1+x)^k$, and $B_0, B_1, B_2, \ldots$ the positive binomial coefficients in the expansion of $(1+x)^{-k}$, that is

$$A_0 = 1, \qquad A_1 = \frac{k}{1}, \qquad A_2 = \frac{k.k-1}{1.2}, \text{ etc.,}$$

$$B_0 = 1, \qquad B_1 = \frac{k}{1}, \qquad B_2 = \frac{k.k+1}{1.2}, \text{ etc.;}$$

then the calculated values of the components are

$$A_{k,\,n-1} = a^{k(n-1)} + (A_0B_n - A_1B_0)a^{k(n-1)-n}z + (A_0B_{2n} - A_1B_n + A_2B_0)$$
$$\times\, a^{k(n-1)-2n}z^2 + (A_0B_{3n} - A_1B_{2n} + A_2B_n - A_3B_0)a^{k(n-1)-3n}z^3 + \ldots$$

The series for this component as well as for all others is finite, for the expansion of P^k shows that the highest power of a must be smaller than k.

The next component is

$$A_{k,n-2} = B_1 a^{k(n-1)-1} + (A_0 B_{n+1} - A_1 B_1) a^{k(n-1)-(n+1)} z$$
$$+ (A_0 B_{2n+1} - A_1 B_{n+1} + A_2 B_1) a^{k(n-1)-(2n-1)} z^2 + \ldots$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$A_{k,n-h-1} = B_h a^{k(n-1)-h} + (A_0 B_{n+h} - A_1 B_h) a^{k(n-1)-(n+h)} z$$
$$+ (A_0 B_{2n+h} - A_1 B_{n+h} + A_2 B_h) a^{k(n-1)-(2n+h)} z^2 + \ldots$$

where h denotes any of the numbers $0, 1, 2, \ldots\ldots n-1$; and finally the last two components, and the most simple, are;

$$A_{k\,1} = B_{n-2}\, a^{k(n-1)-(n-2)} + (A_0 B_{2n-2} - A_1 B_{n-2}) a^{k(n-1)-(2n-2)} z$$
$$+ (A_0 B_{3n-2} - A_1 B_{2n-2} + A_2 B_{n-2}) a^{k(n-1)-(3n-2)} z^2 + \ldots$$
$$A_{k\,0} = B_{n-1}\, a^{k(n-1)-(n-1)} + (A_0 B_{2n-1} - A_1 B_{n-1}) a^{k(n-1)-(2n-1)} z$$
$$+ (A_0 B_{3n-1} - A_1 B_{2n-1} + A_2 B_{n-1}) a^{k(n-1)-(3n-1)} z^2 + \ldots$$

The successive formation of the above components for $k = 2, 3, 4,$ etc. I will call *linear* algorithms, for the degree of approximation is proportional to the number k.

2. Specializing for $k = 2$, we obtain as components of second order;

$$\left.\begin{array}{l} 1.a^{2n-2} + (n-1)a^{n-2} z; \\ 2\, a^{2n-3} + (n-2)a^{n-3} z; \\ \vdots \qquad\qquad \vdots \\ h\, a^{2n-(h+1)} + (n-h)a^{n-(h+1)} z; \\ \vdots \qquad\qquad \vdots \\ (n-2)a^{n+1} + 2.a^1 z; \\ (n-1)a^n \quad + 1.z; \\ \qquad na^{n-1}. \end{array}\right\} \ldots\ldots\ldots\ldots (3)$$

The ratio of the last two components is the well known method of Newton:

$$a_1 = \frac{(n-1)a^n + z}{na^{n-1}},$$

which was reproduced by Mr. Evans in the ANALYST of January, 1876. Of all the n different values for $\sqrt[n]{z}$, furnished by the components of second order, one will be the best, independent of z and a, and this is the one where $h = \frac{1}{2}(n-1)$; i. e.

$$a_1 = a.\frac{(n-1)a^n + z(n+1)}{(n+1)a^n + z(n-1)}, \quad \ldots\ldots\ldots\ldots (4)$$

where a denotes any convenient initial value, and a_1 the corrected value for $\sqrt[n]{z}$.—The method under (4) is of third order, and reappears among the n methods for $k = 3$.

3. Specializing for $k = 3$, we obtain the components of third order:

$$\left.\begin{array}{l}
1.a^{3n-3} + \frac{n^2+3n-2.2}{2} a^{2n-3} z + \frac{(n-1)(n-2)}{2} a^{n-3} z^2; \\
\frac{3}{1} a^{3n-4} + \frac{n^2+5n-6.2}{2} a^{2n-4} z + \frac{(n-2)(n-3)}{2} a^{n-4} z^2; \\
\frac{3.4}{1.2} a^{3n-5} + \frac{n^2+7n-12.2}{2} a^{2n-5} z + \frac{(n-3)(n-4)}{2} a^{n-5} z^2; \\
\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\frac{h(h+1)}{2} a^{3n-(h+2)} + \frac{n^2+(2h+1)n-h(h+1).2}{2} a^{2n-(h+2)} z \\
\qquad\qquad + \frac{(n-h)(n-h-1)}{2} a^{n-(h+2)} z^2; \\
\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\frac{(n-1)n}{2} a^{3n-(n+1)} + \frac{(n+1)n}{2} a^{2n-(n+1)} z; \\
\frac{(n+1)n}{2} a^{3n-(n+2)} + \frac{(n-1)n}{2} a^{2n-(n+2)} z.
\end{array}\right\} \ldots (5)$$

The last two components furnish the method (4) again after a slight reduction. The method under (4) seems to be the most practical of all, considering its simple form and rapidity of approximation. If the initial value (a) has any number of correct decimals, the next corrected value has three times this number of correct decimals.

4. By fixing any value of k and repeating with any of the n possible methods the same process with the number k, we have n different algorithms of the order k. For $k = 2$ we double with every step the number of correct decimals; for $k = 3$ we multiply the number of correct decimals by 3, and so on:—Our general principle furnishes methods of any required degree of approximation.

If we would avoid raising to high powers, we have to prepare the given number z by proper multiplication so that its value is nearly unity. In this case 1 is a good initial value. Then form all the components of the second order and its n algorithms, and take the arithmetical mean of them. This value will multiply the number of correct decimals of the initial value by four.

A theorem still more general than the one here explained, and numerical examples, I am obliged to suppress here for want of room.

[Dr. Eggers writes under date of May 18th, "The case of a revolving ellipsoid of three unequal axes is treated of in Kirchhof's Vorlesungen uber Mathematische Physik, (Leipzig, editor Teubner, 1876.) Vorlesung 25; and by Dirichlet in Abhandlungen der Koniglichen gesellschaft der Wissenschaften zu Gottingen, volume 8, 1860; and Rankine treats of it in London Philos. Transactions 1863, Part I, p. 227."]